$$\varphi_2' = -2p_0Re^{\varepsilon\varphi_3}, \quad R\varphi_2 + \varphi_1' = 0$$
$$(1 - \varphi_1)\varphi_3' + R\varphi_3'' - R\varphi_2 = 0$$

the solution of which

$$\varphi_i = \varphi_i^{(0)} + \varepsilon\varphi_i^{(1)} + O(\varepsilon^2), \quad i = 1, 2, 3$$
$$\varphi_1^{(0)} = P_0(R^4/4 - R^2/2), \quad \varphi_2^{(0)} = p_0(1 - R^2), \quad \varphi_3^{(0)} =$$
$$p_0\left(\int_R^1\left(\int_0^R (1 - R^2) RF(R)\,dR\right)\frac{dR}{RF(R)} + \alpha\right), \quad \varphi_1^{(1)} = -\int_R^1 R\varphi_2^{(1)}dR$$
$$\varphi_2^{(1)} = -2p_0\int_R^1 R\varphi_3^{(0)}dR, \quad \varphi_3^{(1)} = \int_R^1\left(\int_0^R (R\varphi_2^{(1)} + \varphi_1^{(1)}\varphi_3^{(0)\prime}) RF(R)\,dR\right)\frac{dR}{RF(R)}$$
$$F(R) = \exp((p_0/4)(R^2 - R^4/4))$$

describes a flow in a tube with permeable walls for a constant rate of injection (suction) $v_{R=1} = -p_0/4$.

For arbitrary functions f and g an invariant solution of the operator X_2 may be written in the form

$$v = -\frac{1}{4}\frac{g'}{g}\varphi(\xi), \quad u = \frac{\varphi(\xi)}{R^2}, \quad T = A_1\ln\xi + A_2$$
$$\varphi = \xi^{1/2}\left(1/_4\int\xi^{-3/2}f(T)\,d\xi + A_3\right), \quad \xi = R^4g(z)$$

where A_i $(i = 1, 2, 3)$ are arbitrary constants, which may be adjusted so that

$$\varphi(\beta_i) = 0, \quad \beta_i > 0, \quad i = 1, 2$$

This solution corresponds to a flow in an annular channel, the radius of the walls of which varies as $R_i = (\beta_i/g)^{1/4}$. Since $g(z)$ is an arbitrary function and the initial system of equations is invariant under shifts in z, we may choose a function $g(z)$ and a range of variation of z such that R_i is practically constant.

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DIFFRACTION OF SHEAR WAVES BY AN ELASTIC CYLINDRICAL INCLUSION WITH TWO CUTS ON THE PHASE BOUNDARY*

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A method /1/ similar to that used in the case of one cut /2/ is used to determine the stress and deformation at the boundary of a cylindrical inclusion with two cuts placed on the contact contour. The external perturbation varies sinusoidally and is a plane wave in an isotropic medium. At the boundary of the inclusion the shear wave is reflected as a shear wave.

1. Formulation of the problem. Using a cylindrical system of coordinates we consider the effect of a plane shear wave on an elastic inclusion in the form of a circular cylinder $r \leqslant a$, $z \in (-\infty, \infty)$, bonded elastically along the edge $r = a$, $\theta \in \Omega = (\alpha_1, \pi - \alpha_2) \cup (\pi + \alpha_2, 2\pi - \alpha_1)$, $z \in (-\infty, \infty)$ where the area $r = a$, $\theta \in \Omega_0$, $(\Omega_0 = [-\alpha_1, \alpha_1] \cup [\pi - \alpha_2, \pi + \alpha_2])$ corresponds to two cuts

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(Fig.1). To simplify the calculations, the cuts are located symmetrically with respect to the incident wave front.

For antiplane deformation, in the case of the above load, the only non-zero component of the displacement $U_z = W$ satisfies the following wave equation:

$$(\Delta + \beta^2) W = 0, \quad r > a \quad (\Delta + \beta_1^2) W = 0, \quad r < a \tag{1.1}$$

where $\beta = w/c$ and $\beta_1 = w/c_1$ are wave numbers.

At the boundary of the inclusion with the medium the following conditions should be satisfied

$$\begin{gathered} W^+ + W^- = W; \quad \sigma^+ + \sigma^- = \sigma, \quad r = a, \quad \theta \in \Omega \\ \sigma^+ + \sigma^- = 0, \quad \sigma = 0, \quad r = a, \quad \theta \in \Omega_0 \end{gathered} \tag{1.2}$$

where $\sigma^\pm$ is the stress corresponding to the shear and the plus sign denotes a component of the incident wave and a minus denotes a component of the reflected wave. Infinite incident and reflected waves should satisfy the radiation condition and the Meixner condition at the edge /3/. Thus, taking into account the symmetry of the problem, we shall look for a solution of Eq.(1.1) of the form (the time factor e^{-iwt} is omitted):

$$W^+ + W^- = \sum_{n=0}^{\infty} \left[\varepsilon_n W_0 \frac{J_n(\beta r)}{J_n(\beta \alpha)} + A_n \frac{H_n(\beta r)}{H_n(\beta a)} \right] \cos n\theta \tag{1.3}$$

$$\varepsilon_n = \begin{cases} 1, & n = 0 \\ 2(-r)^n, & n = 1, 2, \ldots, \infty \end{cases}$$

A solution of the second equation of (1.1) should be bounded at the initial coordinate and have finite energy; thus, we have

$$W = \sum_{n=0}^{\infty} B_n \frac{J_n(\beta_1 r)}{J_n(\beta_1 a)} \cos n\theta. \tag{1.4}$$

2. *Solution.* Representing the stress in terms of the displacement and substituting the boundaries conditions (1.2) we obtain a dual series equations. To solve these, we use a technique for solving a special class of dual equations /1/. For $\theta \in \Omega$, we add the terms

$$-a\mu^{-1} h(\theta) - i\beta a W_0 \cos \eta\theta \exp(i\beta a \cos\theta) \tag{2.1}$$

to the left-hand side of the third condition of (1.2).

Here, $h(\theta)$ is an unknown function to the shear stress on the edge $r = a$, $\theta \in \Omega$, corresponding to the field of the perturbed wave, μ is the shear modulus for the medium and μ_1 is that for the inclusion. The dual equations derived by adding terms to (2.1) are given as follows:

$$A_0 r_0 + \sum_{n=1}^{\infty} n A_n \cos n\theta + \sum_{n=1}^{N\to\infty} A_n \left[\frac{\beta a H_{n+1}(\beta a)}{H_n(\beta a)} - 2n \right] \cos n\theta = \Lambda(\theta) - i\beta a W_0 \cos\theta \exp(-i\beta a \cos\theta) \tag{2.2}$$

$$B_0 \rho_0 + \sum_{n=1}^{\infty} n B_n \cos n\theta + \sum_{n=1}^{N\to\infty} B_n \left[\frac{\beta_1 a J_{n+1}(\beta_1 a)}{J_n(\beta_1 a)} - 2n \right] \cos n\theta = \Lambda_1(\theta)$$

$$\Lambda(\theta) = \begin{cases} 0, & \theta \in [0, \alpha_1] \cup [\pi - \alpha_2, \pi] \\ -a\mu^{-1} h(\theta), & \theta \in (\alpha_1, \pi - \alpha_2) \end{cases}$$

The function $\Lambda_1(\theta)$ corresponds to the substitution $\mu \to \mu_1$. The coefficients A_n and B_n are found from Fourier series expansions of the right-hand sides of Eqs.(2.1) and are given by:

$$A_0 \tau_0 = -\beta a W_0 J_1(\beta a) - \frac{a}{\mu\pi} \int_{\alpha_1}^{\pi - \alpha_1} h(\theta)\, d\theta$$

$$A_n = 2a w_n \left[\frac{1}{\mu n \pi} \int_{\alpha_1}^{\pi - \alpha_1} h(t) \cos nt\, dt - \frac{\beta W_0 (-i)^n J_n'(\beta a)}{n} \right]$$

$$w_n = \begin{cases} 1 - \tau_n, & n = 1, 2, \ldots, N \\ 1, & n > N \end{cases}; \quad \tau_n = \frac{\beta a H_{n+1}(\beta a) - 2n H_n(\beta a)}{\beta a H_{n+1}(\beta a) - n H_n(\beta a)}$$

$$\beta_0 \rho_0 = -\frac{a}{\mu_1 \pi} \int_{\alpha_1}^{\pi - \alpha_1} h(\theta)\, d\theta, \quad \beta_n = \frac{-2a\xi_n}{\mu_1 n \pi} \int_{\alpha_1}^{\pi - \alpha_1} h(t) \cos nt\, dt.$$

ξ_n is a function like w_n, where τ_n has the form given in formula (2.3) for ρ_n (see below).

Inserting the coefficients A_n and B_n obtained into condition (2.2) and changing the order of summation and integration, we obtain an integral equation with a discontinuous kernel

$$\frac{2a}{\pi}\left[\gamma\int_{\alpha_1}^{\pi-\alpha_2} h(t)\,\eta_n(t,\theta)\,dt-\xi_n\int_{\alpha_1}^{\pi-\alpha_2} h(t)\cos ntdt\right]=A_0-B_0+W(P_n+\theta_n) \tag{2.3}$$

$$\gamma=\frac{1}{\mu}-\frac{1}{\mu_1},\quad \eta_n(t,\theta)=\sum_{n=1}^{\infty}\frac{\cos nt\cos n\theta}{n},\quad G=\frac{\mu}{\mu_1}$$

$$\xi_n=\frac{1}{\mu}\sum_{n=1}^{N}\frac{\rho_n-G\tau_n}{nG},\quad \rho_n=\frac{\beta_1 aJ_{n+1}(\beta_1 a)-2nJ_n(\beta_1 a)}{\beta_1 aJ_{n+1}(\beta_1 a)-nJ_n(\beta_1 a)}$$

$$Q_n=\sum_{n=1}^{N}(-i)^n\frac{\tau_n}{n}J_n(\beta a)\cos n\theta,$$

$$P_n=J_0(\beta\alpha)\sum_{n=1}^{\infty}(-i)^n\left[\frac{\beta a}{n}J_n'(\beta a)+J_n(\beta a)\right]\cos n\theta$$

We introduce new variables ζ and φ according to the formulae

$$\cos t=b+b_1\cos\zeta,\quad \cos\theta=b+b_1\cos\varphi \tag{2.4}$$

where b and b_1 are constants such that the variables φ and ζ completely cover the range from 0 to π. They are determined in the form

$$b=(\cos\alpha_1-\cos\alpha_2)/2,\quad b_1=(\cos\alpha_1+\cos\alpha_2)/2$$

From Eq.(2.4), we have

$$\alpha\theta/\alpha\varphi=\Theta(\theta),\quad \Theta(\theta)=(\cos\alpha_1-\cos\theta)^{1/2}(\cos\alpha_2+\cos\theta)^{1/2}/2\sin\theta \tag{2.5}$$

The function $h(\theta)\,\alpha\theta/\alpha\varphi$ satisfies the Dirichlet condition in the interval from 0 to π; thus, it has a Fourier series expansion

$$h(\theta)\frac{d\theta}{d\varphi}=\sum_{m=1}^{\infty}d_m\cos(m-1)\varphi,\quad \varphi=\arccos\left(\frac{\cos\theta-b}{b_1}\right) \tag{2.6}$$

Taking into account (2.5), we have

$$h(\theta)\,|_{\theta\in(d_1,\,\pi-\alpha_2)}=\sum_{m=1}^{\infty}\alpha_m\frac{\cos(m-1)\varphi}{\Theta(\theta)}$$

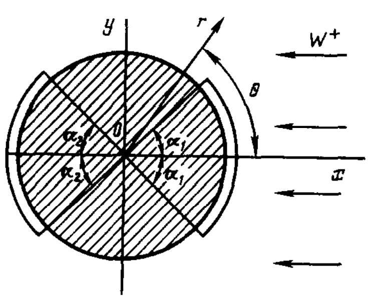

Fig.1

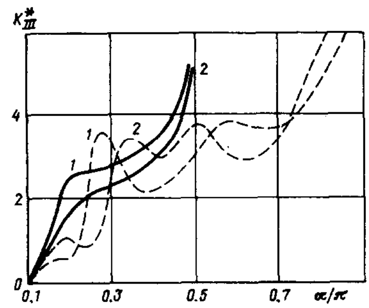

Fig.2

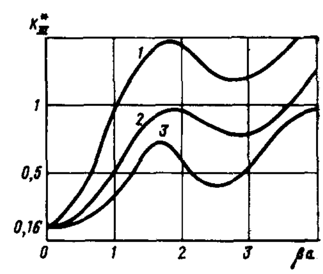

Fig.3

Substituting expression (2.6) into Eq.(2.3) and using the orthogonality of the functions, we obtain a system of finite algebraic equations in the unknowns $d_j(m)$:

$$b_m{}^1\alpha_m+\frac{1}{\mu}\sum_{j=1}^{N}\alpha_j D_{jm}=\frac{2W_0}{a}(P_m-\theta_m)+C_0\delta_m \tag{2.7}$$

where b_m^1, D_{jm}, C_0 and δ_m are the same as in /2/.

3. Special cases. We will determine the stress intensity factor for a crack in the illuminated zone

$$K_{III}^1 = \lim_{\theta \to \alpha_1} [\sqrt{2\pi a (\theta - \alpha_1)}\, h(\theta)] = \sqrt{\pi a \operatorname{tg} \alpha_1} \sum_{m=1}^{\infty} \alpha_m$$

For cracks in the shadow zone, we have

$$K_{III}^2 = \lim_{\theta \to \alpha_2} [\sqrt{2\pi a (\theta - \pi + \alpha_2)}\, h(\theta)] = \sqrt{\pi a \operatorname{tg} \alpha_2} \sum_{m=1}^{\infty} (-1)^m \alpha_m$$

To compare the results, we consider the case when $\alpha_2 = 0,\ \alpha_1 \neq 0,$ when the stress intensity factor is as in /2/. For $\alpha_2 \neq 0,\ \alpha_1 = 0$ we have the case of a shadow crack when the stress intensity factor has the form

$$K_{III} = \sqrt{\pi \alpha \operatorname{tg} \frac{\alpha_2}{2}} \sum_{h=1}^{\infty} (-1)^h \alpha_h$$

To determine the static stress intensity factor, we use asymptotic cylindrical functions with small values of the argument. From system (2.7), we find (for $N = 0$)

$$\alpha_1 = \frac{C_0}{b_1{}^1}, \quad C_0 = \frac{W_0}{a}\left[J_0(\beta a) - \frac{J_1(\beta a) H_0(\beta a)}{H_1(\beta a)}\right]$$

$$b_1{}^1 = \gamma \ln b_1 + \frac{H_0(\beta a)}{\mu \beta a H_1(\beta a)} - \frac{J_0(\beta_1 a)}{\mu_1 \beta_1 a J_1(\beta_1 a)}$$

In the limit we obtain

$$\lim_{\beta\alpha \to 0} \alpha_1 = 2\mu\mu_1 \frac{W_0}{\alpha}$$

Then the static stress intensity factor is determined by the expression

$$K_0{}^m = 2\mu\mu_1 \sqrt{\pi a \operatorname{tg} \alpha_m} W_0/a$$

where $m = 1$ for an illuminated crack and $m = 2$ for an shadow crack.

4. Numerical results. Fig.2 shows a graph of the calculated values of the stress intensity factor for a constant wave number for cracks in various positions, against the crack length. Here, curve *1* corresponds to an illuminated crack and curve *2* to a shadow, crack, the dashed lines represent the case of a single crack and the continuous lines that of two cracks. As distinct from the behaviour of the dynamic stress intensity factor, in the static case, the graphs are characterized by the presence of local maxima and minima. These two point to the hypothesis that there is a distinctive resonance for a given crack length and load frequency.

If there is a single crack on the contact contour, located in the unilluminated zone, the relative stress intensity factor is given by $K_{III}^* = |K_{III}/K_0|$, which is shown in Fig.3 for a constant crack length $\alpha = \pi/4$ as a function of the wave number, for various contact characteristics. Here, the following notations is used for the curves: *1*-rigid contact, *2*-elastic contact with $\mu_1/\mu = 20$ and $\rho_1/\rho = 2$ and *3*-elastic in the case of a uniform medium. In this case, phenomena analogous to those mentioned in the previous example are observed.

Using the local fracture criterion /4/ for this case we obtain the local fracture criterion

$$\gamma = (1/\mu_1 + 1/\mu)\, K_{III}^{*}/8$$

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